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Full Length Research Paper

A study of Green's functions for three-dimensional problem in thermoelastic diffusion media

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The purpose of the present paper is to study the three-dimensional general solution and Green's functions in transversely isotropic thermoelastic diffuson media for static problem. With this objective, two displacement functions are introduced to simplify the basic equation and a general solution is then obtained by using the operator theory. Based on the obtained general solution, the three- dimensional Green's functions for a study point heat source on the apex of a transversely isotropic thermoelastic cone are constructed by four newly introduced harmonic functions. The components of displacement, stress, temperature distribution and mass concentration are expressed in terms of elementary functions and are convenient to use. When the apex angle 2α equals to π, then we obtain the solution for semi-infinite body with a surface point. From the present investigation, a special case of interest is deduced to depict the effect of diffusion on components of stress and temperature distribution.

Key words: Thermoelastic diffuson media, Green's function, transversely isotropic.

INTRODUCTION

Fundamental solutions or Green's functions play an important role in the solution of numerous problems in the mechanics and physics of solids. Green's functions can be used to construct many analytical solutions of boundary value problems. They are essential in boundary element method as well as the study of cracks, defects and inclusion. They are a basic building block of future works. For example, fundamental solutions can be used to construct many analytical solutions of practical problems when boundary conditions are imposed. Ding et al. (1996) derived the general solutions for coupled equations in piezoelectric media. Dunn and Wienecke (1999) investigated the half space Green's functions in transversely isotropic piezoelectric solid. Pan and Tanon (2000) studied the Green's functions for three dimensional problems in anisotropic piezoelectric solids.

When thermal effects are considered, Sharma (1958) investigated the fundamental solution in transversely isotropic thermoelastic material in an integral form. Chen et al. (2004) derived the three dimensional general solution in transversely isotropic thermoelastic materials. Hou et al. (2008, 2009) investigated the Green's function for two and three-dimensional problem for a steady point heat source in the interior of a semi-infinite thermoelastic material. Also, Hou et al. (2011) investigated the two dimensional general solutions and fundamental solutions in orthotropic thermoelastic materials.

Diffusion can be defined as random walk of assembly

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of particles from a high concentration region to a low concentration region. An example of diffusion is heat transport or movement transport. Thermal diffusion utilizes the transfer of heat across a thin liquid or gas to accomplish isotope separation. Today, thermoelasticity remains a practical process to separate isotopes of noble gases (e.g. xexon) and other light isotopes (e.g. carbon) for research purposes.

Nowacki (1974a, b, c, d) developed the theory of thermoelastic diffusion by using coupled thermoelastic model. Sherief and Saleh (2005) developed the generalized theory of thermoelastic diffusion with one relaxation time which allows finite speeds of propagation of waves. Kumar and Kansal (2008) derived the basic equations for generalized thermoelastic diffusion (G-L model) and discussed the Lamb waves. When diffusion effects are considered, Kumar and Chawla (2011a) derived the Fundamental solution in orthotropic thermoelastic diffusion material. Kumar and Chawla (2011b) discussed the plane wave propagation in the context of anisotropic three-phase-lag and two-phase-lag model of thermoelasticity. Kumar and Chawla (2012) derived the Green's functions for two-dimensional problem in orthotropic thermoelastic diffusion media. Recently, Kumar and Chawla (2013) discussed the problem of reflection and transmission in thermoelastic media with three-phase-lag model. However, the important Green's function for three-dimensional problem function in transversely isotropic thermoelastic diffuson material has not been discussed so far.

Keeping in view of these applications, the three dimensional general solution and Green's function in transversely isotropic thermoelastic diffuson elastic medium for steady state problem was studied. After applying the dimensionless quantities and using the operator theory, the general expression for displacement components, mass concentration and temperature change are derived in terms of four harmonic functions. By virtue of the obtained general solution, the three-dimensional Green's functions for a study point heat source on the apex of a transversely isotropic thermoelastic cone are constructed by four newly introduced harmonic functions. From the present investigation, a special case of interest is also deduced to depict the effect of diffusion.

Basic equations

Following Sherief and Saleh (2005) the basic governing equations for homogenous anisotropic generalized thermoelastic diffusion solid in the absence of body forces, heat and mass diffusion sources are:

(1) Constitutive relations:

$$\sigma_{ij} = c_{ijkm}\varepsilon_{km} + a_{ij}\mathrm{T} + b_{ij}C \qquad (1)$$

(2) Equations of motion:

$$c_{ijkm}\varepsilon_{km,j} + a_{ij}\mathrm{T}_{,j} + b_{ij}C_{,j} = \rho\ddot{u}_i \qquad (2)$$

(3) Equation of heat conduction:

$$\rho C_E\dot{\mathrm{T}} + a\mathrm{T}_0\dot{C} - a_{ij}\mathrm{T}_0\dot{\varepsilon}_{ij} = K_{ij}\mathrm{T}_{,ij} \qquad (3)$$

(4) Equation of mass diffusion:

$$-\alpha_{ij}{}^{*}b_{km}\varepsilon_{km,ij} - \alpha_{ij}{}^{*}bC_{,ij} + \alpha_{ij}{}^{*}a\mathrm{T}_{,ij} = -\dot{C} \qquad (4)$$

Here, $c_{ijkm}(=c_{kmij}=c_{jikm}=c_{ijmk})$ are elastic parameters; $a_{ij}(=a_{ji}),\quad b_{ij}(=b_{ji})$ are respectively, the tensor of thermal and diffusion moduli. ρ is the density and C_E is the specific heat at constant strain, a,b are respective coefficients describing the measure of thermoelastic diffusion effects and of diffusion effects, T_0 is the reference temperature assumed to be such that $\left|\frac{T}{T_0}\right| << 1.$ $K_{ij}(=K_{ji}),\sigma_{ij}(=\sigma_{ji})$ and $\varepsilon_{ij} = \frac{u_{i,j}+u_{j,i}}{2}$ denote the components of thermal conductivity, stress and strain tensor respectively. $T(x,y,z,t)$ is the temperature change from the reference temperature T_0 and C is the mass concentration. u_i is a component of displacement vector while $\alpha^{*}_{ij}(=\alpha^{*}_{ji})$ are diffusion parameters.

In the above equations, the symbol (,) followed by a suffix denotes differentiation with respect to spatial coordinate and a superposed dot (".") denotes the derivative with respect to time respectively.

Following Slaughter (2002), applying the transformation, we have:

$$x' = x\cos\phi + y\sin\phi, \quad y' = -x\sin\phi + y\cos\phi,\ z' = z, \qquad (5)$$

Where ϕ is the angle of rotation in the $x-z$ plane. In the Equations (1) to (4), the stress-strain-temperature-concentration relation, equations of motion, heat conduction and mass diffusion equation in homogeneous, transversely isotropic thermoelastic diffusion media in cartesian coordinates (x,y,z) can be written as:

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{yz}\\ \sigma_{xz}\\ \sigma_{xy}\end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0\\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0\\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & c_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & c_{44} & 0\\ 0 & 0 & 0 & 0 & 0 & c_{66}\end{bmatrix}\begin{bmatrix}e_{xx}\\ e_{yy}\\ e_{zz}\\ 2e_{yz}\\ 2e_{xz}\\ 2e_{xy}\end{bmatrix} - \begin{bmatrix}a_1\\ a_1\\ a_3\\ 0\\ 0\\ 0\end{bmatrix}T - \begin{bmatrix}b_1\\ b_1\\ b_3\\ 0\\ 0\\ 0\end{bmatrix}C, \qquad (6)$$

$$c_{11}\frac{\partial^2 u}{\partial x^2}+c_{66}\frac{\partial^2 u}{\partial y^2}+c_{44}\frac{\partial^2 u}{\partial z^2}+(c_{12}+c_{66})\frac{\partial^2 v}{\partial x\partial y}+(c_{13}+c_{44})\frac{\partial^2 w}{\partial x\partial z}-a_1\frac{\partial T}{\partial x}-b_1\frac{\partial C}{\partial x}=\rho\frac{\partial^2 u}{\partial t^2}, \tag{7}$$

$$(c_{12}+c_{66})\frac{\partial^2 u}{\partial x\partial y}+c_{66}\frac{\partial^2 v}{\partial x^2}+c_{11}\frac{\partial^2 v}{\partial y^2}+c_{44}\frac{\partial^2 v}{\partial z^2}+(c_{13}+c_{44})\frac{\partial^2 w}{\partial y\partial z}-a_1\frac{\partial T}{\partial y}-b_1\frac{\partial C}{\partial y}=\rho\frac{\partial^2 v}{\partial t^2}, \tag{8}$$

$$(c_{13}+c_{44})\frac{\partial^2 u}{\partial x\partial z}+(c_{13}+c_{44})\frac{\partial^2 v}{\partial y\partial z}+c_{44}\frac{\partial^2 w}{\partial x^2}+c_{44}\frac{\partial^2 w}{\partial y^2}+c_{33}\frac{\partial^2 w}{\partial z^2}-a_3\frac{\partial T}{\partial z}-b_3\frac{\partial C}{\partial z}=\rho\frac{\partial^2 w}{\partial t^2}, \tag{9}$$

$$\rho C_E\frac{\partial T}{\partial t}+aT_0\frac{\partial C}{\partial t}+T_0\left(a_1\left(\frac{\partial \dot{u}}{\partial x}+\frac{\partial \dot{v}}{\partial y}\right)+a_3\frac{\partial \dot{w}}{\partial z}\right)=\left[K_1\left(\frac{\partial}{\partial x^2}+\frac{\partial}{\partial y^2}\right)T+K_3\frac{\partial T}{\partial z^2}\right]. \tag{10}$$

$$b_1\left[\alpha_1^*\left(\frac{\partial^3}{\partial x^3}+\frac{\partial^3}{\partial x\partial y^2}\right)u+\alpha_3^*\frac{\partial^3 u}{\partial x\partial z^2}\right]+b_1\left[\alpha_1^*\left(\frac{\partial^3}{\partial x^2\partial y}+\frac{\partial^3}{\partial y^3}\right)v+\alpha_3^*\frac{\partial^3 v}{\partial y\partial z^2}\right]+b_3\left[\alpha_1^*\left(\frac{\partial^3}{\partial x^2\partial z}+\frac{\partial^3}{\partial y^2\partial z}\right)w+\alpha_3^*\frac{\partial^3 w}{\partial z^3}\right]+a\left[\alpha_1^*\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)T+\alpha_3^*\frac{\partial^2 T}{\partial z^2}\right]-b\left[\alpha_1^*\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)C+\alpha_3^*\frac{\partial^2 C}{\partial z^2}\right]=-\frac{\partial C}{\partial t} \tag{11}$$

Where

$a_{ij}=-a_i\delta_{ij},\quad b_{ij}=-b_i\delta_{ij},\quad K_{ij}=K_i\delta_{ij},$ i is not summed and $c_{66}=\dfrac{c_{11}-c_{12}}{2}.$

FORMULATION OF THE PROBLEM

We consider a homogenous transversely isotropic thermoelastic diffusion medium. Let us take Oxyz as the frame of reference in Cartesian coordinates.

For three dimensional problems, we assume the displacement vector, temperature distribution and mass concentration are respectively, of the form:

$$\vec{u}=(u,v,w),\quad \mathrm{T}(x,y,z,t),\quad C(x,y,z,t). \tag{12}$$

Moreover, we are discussing steady problem

$$\frac{\partial u}{\partial t}=\frac{\partial v}{\partial t}=\frac{\partial w}{\partial t}=\frac{\partial T}{\partial t}=\frac{\partial C}{\partial t}=0. \tag{13}$$

We define the dimensionless quantities as:

$$\left(x',y',z',u',v',w',b',r'\right)=\frac{\omega_1^*}{v_1}\left(x,y,z,u,v,w,b,r\right),$$

$$\left(T',C'\right)=\frac{1}{c_{11}}\left(a_1T,b_1C\right),$$

$$\sigma_{ij}'=\frac{\sigma_{ij}}{a_1\mathrm{T}_0},\ H'=\frac{a_1}{c_{11}K_3}H,$$

Where

$$v_1^2=b_1,\quad \omega_1^*=\frac{aC_{11}}{K_1}. \tag{14}$$

Applying the dimensionless quantities defined by Equation (14) in Equations (7) to (11), after suppressing the primes, we obtain:

$$\left(\frac{\partial^2}{\partial x^2}+\delta_2\frac{\partial^2}{\partial y^2}+\delta_1\frac{\partial^2}{\partial z^2}\right)u+\left(\delta_3\frac{\partial^2}{\partial x\partial y}\right)v+\left(\delta_4\frac{\partial^2}{\partial x\partial z}\right)w-\left(\frac{\partial}{\partial x}\right)T-\left(\frac{\partial}{\partial x}\right)C=0, \tag{15}$$

$$\left(\delta_3\frac{\partial^2}{\partial x\partial y}\right)u+\left(\delta_2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\delta_1\frac{\partial^2}{\partial z^2}\right)v+\left(\delta_4\frac{\partial^2}{\partial z\partial y}\right)w-\left(\frac{\partial}{\partial y}\right)T-\left(\frac{\partial}{\partial y}\right)C=0 \tag{16}$$

$$\left(\delta_4\frac{\partial^2}{\partial x\partial z}\right)u+\left(\delta_4\frac{\partial^2}{\partial z\partial y}\right)v+\left(\delta_1\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+\delta_5\frac{\partial^2}{\partial z^2}\right)w-\varepsilon_1\left(\frac{\partial}{\partial z}\right)T-\gamma_1\left(\frac{\partial}{\partial z}\right)C=0, \tag{17}$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)T+\varepsilon_2\left(\frac{\partial^2}{\partial z^2}\right)T=0, \tag{18}$$

$$\frac{\partial}{\partial x}\left[q_1^*\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+q_2^*\frac{\partial^2}{\partial z^2}\right]u+\frac{\partial}{\partial y}\left[q_1^*\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+q_2^*\frac{\partial^2}{\partial z^2}\right]v+\frac{\partial}{\partial z}\left[q_3^*\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+q_4^*\frac{\partial^2}{\partial z^2}\right]w+\left[q_5^*\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+q_6^*\frac{\partial^2}{\partial z^2}\right]T-\left[q_7^*\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+q_8^*\frac{\partial^2}{\partial z^2}\right]C=0, \tag{19}$$

Where

$$\left(\delta_1,\delta_2,\delta_3,\delta_4,\delta_5\right)=\frac{1}{c_{11}}\left(c_{44},c_{66},c_{12}+c_{66},c_{13}+c_{44},c_{33}\right),\ \varepsilon_1=\frac{a_3}{a_1},\ \gamma_1=\frac{b_3}{b_1},\ \varepsilon_2=\frac{K_3}{K_1},$$

$$\left(q_1^*,q_2^*,q_3^*,q_4^*\right)=\frac{1}{c_{11}}\left(\alpha_1^*\omega_1^*b_1,\alpha_3^*\omega_1^*b_1,\alpha_1^*\omega_1^*b_3,\alpha_3^*\omega_1^*b_3\right),\left(q_5^*,q_6^*\right)=\frac{1}{a_1}\left(\alpha_1^*\omega_1^*a,\alpha_3^*\omega_1^*a\right),$$

$$\left(q_7^*,q_8^*\right)=\frac{1}{b_1}\left(\alpha_1^*\omega_1^*b,\alpha_3^*\omega_1^*b\right),$$

$$a_1=(c_{11}+c_{12})\alpha_1+c_{13}\alpha_{3,},\ a_3=2c_{13}\alpha_1+c_{33}\alpha_3,\ b_1=(c_{11}+c_{12})\alpha_{1c}+c_{13}\alpha_{3c},$$

$$b_3=2c_{13}\alpha_{1c}+c_{33}\alpha_{3c},c_{66}=\frac{c_{11}-c_{12}}{2}.$$

STATIC GENERAL SOLUTIONS

Two displacements functions Ψ and G are introduced as follows:

$$u=\frac{\partial\Psi}{\partial y}-\frac{\partial G}{\partial x}, v=-\frac{\partial\Psi}{\partial x}-\frac{\partial G}{\partial y} \tag{20}$$

Using the displacements functions Ψ and G in Equations (15) - (19), we obtain

$$\left[\delta_2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+\delta_1\frac{\partial^2}{\partial z^2}\right]\Psi=0, \tag{21}$$

$$D\begin{Bmatrix}G\\w\\C\\T\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\\0\end{Bmatrix} \tag{22}$$

where D is the differential operator matrix given by

$$\begin{bmatrix}\Delta+\delta_1\frac{\partial^2}{\partial z^2} & -\delta_4\frac{\partial}{\partial z} & 1 & 1\\ -\delta_4\Delta\frac{\partial}{\partial z} & \delta_1\Delta+\delta_5\frac{\partial^2}{\partial z^2} & -\gamma_1\frac{\partial}{\partial z} & -\varepsilon_1\frac{\partial}{\partial z}\\ -\left(q_1^*\Delta^2+q_2^*\Delta\frac{\partial^2}{\partial z^2}\right) & q_3^*\Delta\frac{\partial}{\partial z}+q_4^*\frac{\partial 3}{\partial z^3} & -\left(q_7^*\Delta+q_8^*\frac{\partial^2}{\partial z^2}\right) & q_5^*\Delta+q_6^*\frac{\partial^2}{\partial z^2}\\ 0 & 0 & 0 & \Delta+\varepsilon_3\frac{\partial^2}{\partial z^2}\end{bmatrix}$$

Equation (22) is a homogeneous set of differential equations in G, w, T, C. The general solution by the operator theory is as follows:

$$G=A_{i1}F, \quad w=A_{i2}F, \quad C=A_{i3}F, \quad T=A_{i4}F \quad (i=1,2,3,4). \tag{23}$$

The determinant of the matrix D is given as:

$$|D|=\left(\bar{a}\frac{\partial^6}{\partial z^6}+\bar{b}\Delta\frac{\partial^4}{\partial z^4}+\bar{c}\Delta^2\frac{\partial^2}{\partial z^2}+\bar{d}\Delta^3\right)\times\left(\Delta+\varepsilon_3\frac{\partial^2}{\partial z^2}\right), \tag{24}$$

Where $\bar{a},\bar{b},\bar{c},\bar{d}$ and Δ are given in Appendix A. The function F in Equation (23) satisfies the following homogeneous equation:

$$|D|F=0 \tag{25}$$

It can be seen that if $i=1,2,3$ are taken in Equation (23), three general solutions are obtained in which $T=0$. These solutions are identical to those without thermal fact and are not discussed here. Therefore if $i=4$ should be taken in Equation (23), the following solution is obtained:

$$u=\frac{\partial\Psi}{\partial y}-\left(\bar{a}_1\Delta^2+\bar{b}_1\Delta\frac{\partial^2}{\partial z^2}+\bar{c}_1\frac{\partial^4}{\partial z^4}\right)\frac{\partial F}{\partial x}, \tag{26}$$

$$v=\frac{\partial\Psi}{\partial x}-\left(\bar{a}_1\Delta^2+\bar{b}_1\Delta\frac{\partial^2}{\partial z^2}+\bar{c}_1\frac{\partial^4}{\partial z^4}\right)\frac{\partial F}{\partial y}, \tag{27}$$

$$w=\left(\bar{a}_2\Delta^2+\bar{b}_2\Delta\frac{\partial^2}{\partial z^2}+\bar{c}_2\frac{\partial^4}{\partial z^4}\right)\frac{\partial F}{\partial z}, \tag{28}$$

$$C=\left(\bar{a}_1\Delta^3+\bar{b}_3\Delta^2\frac{\partial^2}{\partial z^2}+\bar{c}_3\Delta\frac{\partial^4}{\partial z^4}+\bar{d}_4\frac{\partial^6}{\partial z^6}\right)F, \tag{29}$$

$$T=\left(\bar{a}\frac{\partial^6}{\partial z^6}+\bar{b}\Delta\frac{\partial^4}{\partial z^4}+\bar{c}\Delta^2\frac{\partial^2}{\partial z^2}+\bar{d}\Delta^3\right)F, \tag{30}$$

Where $\bar{a}_i,\bar{b}_i,\bar{c}_i (i=1,2,3)$ and $\bar{d}_4$ are given in Appendix B.

In cylindrical coordinate $(r,\theta,z),$ the general solution can be easily obtained. In fact, the expression for w,T and C are identical to that in Equations (26) to (31), while those r radial and circumferential displacements u_r and u_θ are, respectively

$$u_r=\frac{\partial\Psi}{r\partial\theta}-\left(\bar{a}_1\Delta^2+\bar{b}_1\Delta\frac{\partial^2}{\partial z^2}+\bar{c}_1\frac{\partial^4}{\partial z^4}\right)\frac{\partial F}{\partial r}, \tag{31}$$

$$u_\theta=-\frac{\partial\Psi}{\partial r}-\left(\bar{a}_1\Delta^2+\bar{b}_1\Delta\frac{\partial^2}{\partial z^2}+\bar{c}_1\frac{\partial^4}{\partial z^4}\right)\frac{\partial F}{r\partial\theta}. \tag{32}$$

Here $\Delta=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$ is the Laplacian in polar coordinates.

The general solutions of Equation (25) in terms of F can be rewritten as:

$$\prod_{j=1}^{4}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z_j^{\,2}}\right)F=0, \tag{33}$$

where

$z_j=s_j z,\ s_4=\sqrt{\frac{K_1}{K_3}}$, and $s_j(j=1,2,3)$ are three roots (with positive real part) of the following algebraic equation

$$\bar{a}s^6-\bar{b}s^4+\bar{c}s^2-\bar{d}=0. \tag{34}$$

As known from the generalized Almansi theorem (Ding et al., 1996) the function F can be expressed in terms of four harmonic functions:

1) $F = F_1 + F_2 + F_3 + F_4$ for distinct $s_j\,(j = 1,2,3,4)$,

2) $F = F_1 + F_2 + F_3 + zF_4$ for $s_1 \neq s_2 \neq s_3 = s_4$,

3) $F = F_1 + F_2 + zF_3 + z^2 F_4$ for $s_1 \neq s_2 = s_3 = s_4$,

4) $F = F_1 + zF_2 + z^2 F_3 + z^3 F_4$ for $s_1 = s_2 = s_3 = s_4$,

where F_j satisfies the following harmonic equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial {z_j}^2}\right) F_j = 0 \qquad (j = 1,2,3,4). \tag{35}$$

The general solution for the case of distinct roots can be derived as follows:

$$u = \frac{\partial \Psi}{\partial y} - \sum_{j=1}^{4} p_{1j} \frac{\partial^5 F_j}{\partial x \partial {z_j}^4}, \qquad v = -\frac{\partial \Psi}{\partial x} - \sum_{j=1}^{4} p_{1j} \frac{\partial^5 F_j}{\partial y \partial {z_j}^4},$$

$$w = \sum_{j=1}^{4} s_j p_{2j} \frac{\partial^5 F_j}{\partial z_j^5}, \quad C = \sum_{j=1}^{4} p_{3j} \frac{\partial^6 F_j}{\partial z_j^6}, \qquad T = p_{44} \frac{\partial^6 F_4}{\partial z_4^6}. \tag{36}$$

Where

$$p_{kj} = \bar{a}_k - \bar{b}_k s_j^2 + \bar{c}_k s_j^4 \quad (k = 1,2)$$

$$p_{3j} = -\bar{a}_3 + \bar{b}_3 s_j^2 - \bar{c}_3 s_j^4 + \bar{d}_4 s_j^6$$

$$p_{44} = -\bar{d} + \bar{c} s_4^2 - \bar{b} s_4^4 + \bar{a} s_4^6$$

In the similar way general solution for the other three cases can be derived. Equation (36) can be further simplified by taking

$$p_{1j} \frac{\partial^4 F_j}{\partial z_j^4} = \psi_j, \qquad (j = 1,2,3,4) \tag{37}$$

and writing $\psi_0 = \psi$.

$$u = \frac{\partial \Psi_0}{\partial y} - \sum_{j=1}^{4} \frac{\partial \psi_j}{\partial x}, v = -\frac{\partial \Psi_0}{\partial x} - \sum_{j=1}^{4} \frac{\partial \psi_j}{\partial y}, \qquad w = \sum_{j=1}^{4} s_j P_{1j} \frac{\partial \psi_j}{\partial z_j}$$

$$C = \sum_{j=1}^{4} P_{2j} \frac{\partial^2 \psi_j}{\partial z_j^2}, \qquad T = P_{34} \frac{\partial^2 \psi_4}{\partial z_4^2}, \tag{38}$$

Where

$$P_{1j} = p_{2j} / p_{1j}, \qquad P_{2j} = p_{3j} / p_{1j}, \qquad P_{34} = p_{44} / p_{14}$$

The function ψ_j satisfies the harmonic equations

$$\left(\Delta + \frac{\partial^2}{\partial {z_j}^2}\right) \psi_j = 0 \qquad j = 0,1,2,3,4. \tag{39}$$

In which

$$z_0 = s_0 z, s_0 = \sqrt{\frac{\delta_2}{\delta_1}}$$

In cylindrical coordinates (r, θ, z), the expression for w, T, C will remain the same as given in Equation (38), while the components of displacement in cylindrical coordinates are

$$u_r = \frac{\partial \Psi_0}{r \partial \theta} - \sum_{j=1}^{4} \frac{\partial \psi_j}{\partial r}, \quad u_\theta = -\frac{\partial \Psi_0}{\partial r} - \sum_{j=1}^{4} \frac{\partial \psi_j}{r \partial \theta}. \tag{40}$$

Introducing the following notations for the components both in Cartesian coordinate (x, y, z) and cylindrical coordinate (r, θ, z),

$$\begin{aligned} U &= u + iv = e^{i\theta}(u_r + iu_\theta), \\ \sigma_1 &= \sigma_{xx} + \sigma_{yy} = \sigma_{rr} + \sigma_{\theta\theta}, \\ \sigma_2 &= \sigma_{xx} - \sigma_{yy} + 2i\sigma_{xy} = e^{2i\theta}(\sigma_{rr} - \sigma_{\theta\theta} + 2i\sigma_{r\theta}), \\ \tau_z &= \sigma_{xz} + i\sigma_{yz} = e^{i\theta}(\sigma_{zr} + i\sigma_{z\theta}). \end{aligned}$$

Upon using the notations, the general solution in Equation (38) in the Cartesian coordinate (x, y, z) can be simplified as

$$U = -\Gamma_1\left(i\Psi_0 + \sum_{j=1}^{4} \Psi_j\right), \qquad w = \sum_{j=1}^{4} s_j P_{1j} \frac{\partial \psi_j}{\partial z_j},$$

$$C = \sum_{j=1}^{4} P_{2j} \frac{\partial^2 \psi_j}{\partial z_j^2}, \qquad T = P_{34} \frac{\partial^2 \psi_4}{\partial z_4^2},$$

$$\sigma_1 = 2\sum_{j=1}^{4}\left(c_{66} - r_j s_j^2\right)\Delta \Psi_j, \qquad \sigma_2 = -2c_{66}^{*}\Gamma_1^2\left(i\Psi_0 + \sum_{j=1}^{4} \Psi_j\right),$$

$$\sigma_{zz} = -\sum_{j=1}^{4} r_j \Delta \Psi_j, \qquad \sigma_{zz} = \Gamma_1\left[\sum_{j=1}^{4} s_j r_j \frac{\partial \Psi_j}{\partial z_j} - i s_0 c_{44}^{*} \frac{\partial \Psi_0}{\partial z_0}\right]. \tag{41}$$

Where

$$\Gamma_1 = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y},$$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$ in Cartesian coordinates (x, y, z),

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r} + \frac{\partial^2}{r^2\partial\theta^2}$$ in cylindrical coordinates (r, θ, z),

and

$$r_j = \frac{c_{11}^* + c_{13}^* P_{1j} s_j^2 - c_{11}^* P_{2j} - c_{11}^* P_{34}}{s_j^2} = c_{44}^*(1 - P_{1j}) = -c_{13}^* - c_{33}^* s_j^2 P_{1j} + \varepsilon_1 c_{11}^* P_{34} + \gamma_1 c_{11}^* P_{2j}, \tag{42}$$

$$(c_{11}^*, c_{13}^*, c_{33}^*, c_{44}^*, c_{66}^*) = \frac{1}{a_1 T_0}(c_{11}, c_{13}, c_{33}, c_{44}, c_{66}). \tag{43}$$

For non-torsional axisymmetric problem, $\Psi_0 = 0$ and $\Psi_j\,(j = 1,2,3,4)$ are independent of θ, such that $u_\theta = 0$ and $\sigma_{z\theta} = \sigma_{r\theta} = 0.$

The general solution given by equations in cylindrical coordinate (r, θ, z) can be simplified to the following form:

$$u_r = -\sum_{j=1}^{4}\frac{\partial\psi_j}{\partial r}, w = \sum_{j=1}^{4} s_j P_{1j}\frac{\partial\psi_j}{\partial z_j}, C = \sum_{j=1}^{4} P_{2j}\frac{\partial^2\psi_j}{\partial z_j^2}, \qquad T = P_{34}\frac{\partial^2\psi_4}{\partial z_4^2},$$
$$\sigma_{rr} = 2c_{66}^*\sum_{j=1}^{4}\frac{1}{r}\frac{\partial\psi_j}{\partial r} - \sum_{j=1}^{4} s_j^2 w_j\frac{\partial^2\psi_j}{\partial z_j^2}, \sigma_{\theta\theta} = -2c_{66}^*\sum_{j=1}^{4}\frac{1}{r}\frac{\partial\psi_j}{\partial r} - \sum_{j=1}^{4}(s_j^2 w_j - 2c_{66}^*)\frac{\partial^2\psi_j}{\partial z_j^2},$$
$$\sigma_{zz} = \sum_{j=1}^{4} r_j\frac{\partial^2\psi_j}{\partial z_j^2}, \qquad \sigma_{zr} = \sum_{j=1}^{4} s_j r_j\frac{\partial^2\psi_j}{\partial r\partial z_j}. \tag{44}$$

For torsional axisymmetric problem $\Psi_j = 0 \;\; (j = 1,2,3,4)$, Ψ_0 is independent of θ, so that $u_r = u_z = 0, T = 0, C = 0$ and $\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{zz} = \sigma_{rz} = 0.$

The general solution can be simplified as:

$$u_\theta = -\frac{\partial\Psi_0}{\partial r}, \sigma_{r\theta} = 2c_{66}^*\left(\frac{1}{2}\frac{\partial^2}{\partial z_0^2} - \frac{\partial^2}{\partial r^2}\right)\Psi_0, \sigma_{z\theta} = -s_0 c_{44}^*\frac{\partial^2\Psi_0}{\partial r\partial z_0}. \tag{45}$$

BOUNDARY CONDITIONS OF CONE

We consider a transversely isotropic thermoelastic diffusion cone $\frac{z}{r} \geq \cot\alpha$, where 2α is the apex angle, whose isotropic plane is perpendicular to $z-$axis. At the origin of the coordinate system, the apex is to be taken. At the apex, a concentrated force $P = p_x i + p_y i + p_z k,$ a concentrated moment $M = M_x i + M_y i + M_z k$ and a point heat source H are applied, where i, j, k are three unit vectors of Cartesian coordinates (x, y, z).

In addition, the cone is loaded on the surface with prescribed density of normal heat flux $\vec{q}_n$ and surface forces $X = \bar{X}_r e_r + \bar{X}_\theta e_\theta + \bar{X}_z e_z,$ where e_r, e_ϕ, e_z are three unit vectors of cylindrical coordinates (r, θ, z), which are related to i, j, k by the following relations:

$$e_r = i\cos\theta + j\sin\theta,\; e_\theta = i\sin\theta + j\cos\theta,\; e_z = k. \tag{46}$$

The boundary conditions in cylindrical coordinates on the cone $z/r = \cot\alpha$ are:

$$\sigma_{rr}\cos\alpha - \sigma_{zr}\sin\alpha = \bar{X}_r, \tag{47}$$

$$\sigma_{r\theta}\cos\alpha - \sigma_{\theta z}\sin\alpha = \bar{X}_\theta, \tag{48}$$

$$\sigma_{zr}\cos\alpha - \sigma_{zz}\sin\alpha = \bar{X}_z, \tag{49}$$

$$K_1\frac{\partial T}{\partial r}\cos\alpha - K_3\frac{\partial T}{\partial z}\sin\alpha = \bar{q}_m, \tag{50}$$

$$\frac{\partial C}{\partial r}\cos\alpha - K_3\frac{\partial C}{\partial z}\sin\alpha = \bar{\eta}_m. \tag{51}$$

As shown in Figure 1, when a segment of cone cut off by $z = b$, its global mechanical concentration and thermal equilibrium equations will be:

$$P + \int_0^{2\pi}\int_0^{b\tan\alpha}(\sigma_{zr}e_r + \sigma_{\theta z}e_\theta + \sigma_{zz}e_z)rdrd\theta + \int_0^{2\pi}\int_0^{b}(\bar{X}_r e_r + \bar{X}_\theta e_\theta + \bar{X}_z e_z)dzd\theta\tan\alpha/\cos\alpha = 0, \tag{52}$$

$$M + \int_0^{2\pi}\int_0^{b\tan\alpha}(-b\sigma_{\theta z}e_r + (b\sigma_{zr} - \sigma_{zz})e_\theta + r\sigma_{\theta z}e_z)rdrd\theta + \int_0^{2\pi}\int_0^{b}[-\bar{X}_r e_r + (\bar{X}_r - \bar{X}_z\tan\alpha)e_\theta + \bar{X}_\theta\tan\alpha e_z)z^2 dzd\theta\tan\alpha/\cos\alpha = 0, \tag{53}$$

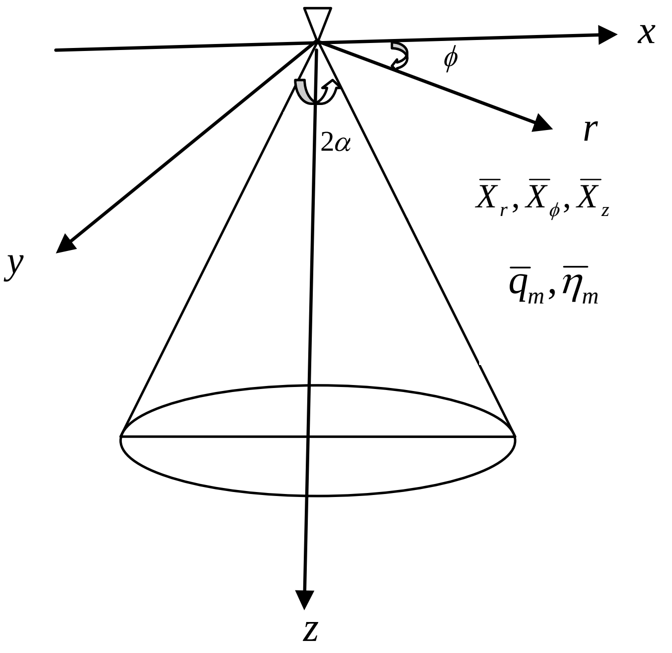

Figure 1. A thermoelastic diffusion cone under loading.

$$\int_0^{2\pi}\int_0^{b\tan\alpha}\frac{\partial T}{\partial z} r dr d\theta + \int_0^{2\pi}\int_0^{b}(K_3\overline{q}_m \sin\alpha + K_1\overline{q}_m\cos\alpha) z dz d\theta \tan\alpha/\cos\alpha = -H, \tag{54}$$

$$\int_0^{2\pi}\int_0^{b\tan\alpha}\frac{\partial C}{\partial z} r dr d\theta + \int_0^{2\pi}\int_0^{b}(\overline{\eta}_m \sin\alpha + \overline{q}_m\cos\alpha) z dz d\theta \tan\alpha/\cos\alpha = 0. \tag{55}$$

Green's function for a point heat source H on the apex of a transversely isotropic thermoelastic diffusion material.

We consider the case only, when point heat source H is applied at the apex and the surface of the cone is traction free, impermeable and thermally insulated, that is,

$$p_x = p_y = p_z = 0,\ M_x = M_y = M_z = 0,$$

$$\overline{X}_r = \overline{X}_\theta = \overline{X}_z = 0, \text{ and } \overline{q}_m = \overline{\eta}_m = 0.$$

The general solution given by Equation (44) is derived in this section.

For non-torsional axisymmetric problem, introduce the following harmonic functions:

$$\Psi_0 = 0 \text{ and } \Psi_j = A_j(z_j \log R_j^* - R_j), \qquad (j = 1,2,3,4) \tag{56}$$

Substituting the values of Ψ_j from Equation (56) in Equation (44), we have:

$$u_r = \sum_{j=1}^{4} A_j \frac{r}{R_j^*}, \qquad w = \sum_{j=1}^{4} s_j P_{1j} A_j \log R_j^*, \qquad C = \sum_{j=1}^{4} P_{2j}\frac{A_j}{R_j}, \tag{57}$$

$$T = P_{34}\frac{A_4}{R_4}, \quad \sigma_{rr} = 2c_{66}^*\sum_{j=1}^{4}\frac{A_j}{R_j^*} - \sum_{j=1}^{4} s_j^2 w_j \frac{A_j}{R_j}, \qquad \sigma_{zz} = \sum_{j=1}^{4} r_j \frac{A_j}{R_j}, \tag{58}$$

$$\sigma_{\theta\theta} = 2c_{66}^*\sum_{j=1}^{4}\frac{A_j}{R_j^*} - \sum_{j=1}^{4}(s_j^2 w_j - 2c_{66}^*)\frac{A_j}{R_j}, \qquad \sigma_{zr} = \sum_{j=1}^{4} s_j r_j A_j \frac{r}{R_j R_j^*}. \tag{59}$$

For non-torsional axisymmetric problem, the boundary condition in Equation (48) has been satisfied, and Equations (49) to (51) can be deduced from the global mechanical, impermeable and thermal equilibrium condition in Equations (52). The only boundary condition in Equation (47) and the following equations need to be satisfied:

$$\int_0^{2\pi}\int_0^{b\tan\alpha}\sigma_{zz} r dr d\theta = 0, \tag{60}$$

$$K_3\int_0^{2\pi}\int_0^{b\tan\alpha}\frac{\partial T}{\partial z} r dr d\theta = -H, \tag{61}$$

$$\int_0^{2\pi}\int_0^{b\tan\alpha}\frac{\partial C}{\partial z} r dr d\theta = 0, \tag{62}$$

Substituting the values of $\sigma_{rr}, \sigma_{zr}, C$ and T from Equation (57) in Equations (47) and (60 to 62) yields

$$\sum_{j=1}^{4} A_j\left(2c_{66}^*\frac{1}{V_j\tan\alpha} - s_j^2 w_j \frac{1}{W_j \tan\alpha} - s_j r_j \frac{1}{W_j V_j}\right) = 0, \tag{63}$$

$$\sum_{j=1}^{4}\frac{r_j}{H_j}A_j = 0, \tag{64}$$

$$\sum_{j=1}^{4}\left(\frac{s_j}{H_j\tan\alpha} - 1\right)s_j P_{2j} A_j = 0, \tag{65}$$

$$\left(\frac{s_4}{H_4\tan\alpha} - 1\right)s_4 P_{24} A_4 = -\frac{H}{2\pi K_3}. \tag{66}$$

Where

$$H_j = \sqrt{1 + s_j^2/\tan^2\alpha},\ N_j = H_j + s_j/\tan\alpha \quad (j = 1,2,3).$$

The constants $A_j\,(j = 1,2,3,4)$ can be determined by solving Equations (63) to (66). When the cone has been reduced to a semi-infinite body, that is, $\alpha = \dfrac{\pi}{2}$ then

$$H_j = N_j = 1 \qquad (j = 1,2,3,4) \tag{67}$$

Using Equation (49) in Equations (45) to (48) can be simplified as:

$$\sum_{j=1}^{4} A_j s_j r_j = 0, \tag{68}$$

$$\sum_{j=1}^{4} r_j A_j = 0, \tag{69}$$

$$\sum_{j=1}^{4} s_j P_{2j} A_j = 0, \tag{70}$$

$$A_4 = \frac{H}{2\pi K_3 s_4 P_{24}} \tag{71}$$

We have determined four constants $A_j (j = 1,2,3)$ from three equations including Equations (68) to (71) by the method of Cramer's rule.

Special case

In the absence of diffusion effects, that is, $b_1 = b_3 = a = b = 0,$ Equations (57) to (59) yields

$$u_r = \sum_{j=1}^{3} A_j \frac{r}{R_j^*}, \qquad u_z = \sum_{j=1}^{3} s_j \hat{P}_{1j} A_j sign(z) \log(R_j^*), \qquad T = \hat{P}_{23} \frac{A_4}{R_4},$$
$$\sigma_{rr} = 2c_{66}^* \sum_{j=1}^{3} \frac{A_j}{R_j^*} - \sum_{j=1}^{3} s_j^2 w_j \frac{A_j}{R_j}, \qquad \sigma_{zz} = \sum_{j=1}^{3} r_j \frac{A_j}{R_j},$$
$$\sigma_{\theta\theta} = 2c_{66}^* \sum_{j=1}^{3} \frac{A_j}{R_j^*} - \sum_{j=1}^{3} (s_j^2 w_j - 2c_{66}^*) \frac{A_j}{R_j}, \quad \sigma_{zr} = \sum_{j=1}^{3} s_j r_j A_j \frac{sign(z) r}{R_j R_j^*}, \tag{72}$$

where s_1, s_2, s_3, s_4 in this case are reduces to s_1, s_2, s_3 with $s_3 = \sqrt{\dfrac{K_1}{K_3}}$ and s_1, s_2 are two roots (with positive real part) of the equation

$$\hat{a} s^4 - \hat{b} s^2 + \hat{c} = 0, \tag{73}$$

and

$$\hat{a} = \breve{\delta}_4 \breve{\delta}_5, \hat{b} = \breve{\delta}_5 + \breve{\delta}_4^2 - \breve{\delta}_1^2, \ \hat{c} = \breve{\delta}_4,$$
$$\hat{P}_{1j} = \frac{\hat{p}_{2j}}{\hat{p}_{1j}}, \breve{P}_{23} = \frac{\hat{p}_{33}}{\hat{p}_{13}},$$
$$\hat{p}_{kj} = \hat{a}_k - \hat{b}_k s_j^2 \ \ (k = 1,2),$$
$$\hat{p}_{33} = \hat{a}_3 - \hat{b}_3 s_j^2 + \hat{c}_3 s_j^4, \hat{a}_1 = -\breve{\delta}_4, \hat{b}_1 = \breve{\delta}_5 - \breve{\delta}_2 \varepsilon_1, \ \ \hat{b}_2 = \breve{\delta}_4 \varepsilon_1,$$
$$\hat{a}_3 = \breve{\delta}_4, \hat{b}_3 = (\breve{\delta}_1^2 - \breve{\delta}_4^2) - \breve{\delta}_5, \ \hat{c}_3 = \breve{\delta}_4 \breve{\delta}_5.$$

Consider the continuity at plane $z = 0$ for u_z and σ_{zr} and substituting the values of σ_{zz}, σ_{zz} and T from Equations (64) with the aid of $s_3 = \sqrt{K_1 / K_3}$ yield the following equations in the absence of diffusion:

$$\sum_{j=1}^{3} s_j \widehat{P}_{1j} A_j = 0. \tag{74}$$

$$\sum_{j=1}^{3} s_j A_j = 0. \tag{75}$$

and

$$A_3 = \frac{H}{2\pi K_3 s_4 P_{24}}$$

The constants $A_j (j = 1,2)$ are determined by two Equations (74) and (75) using the method of Cramer's rule.

The above results are similar as obtained by Hou et al. (2005).

NUMERICAL RESULTS AND DISCUSSION

Here, the numerical discussions are reported and analysis is conducted for magnesium material. Following Dhaliwal and Singh (2005), the values of physical constants are taken as:

$$\mathrm{c}_{11} = 5.974 \times 10^{10} \mathrm{N.m}^{-2}, \mathrm{c}_{12} = 2.624 \times 10^{10} \mathrm{N.m}^{-2}, \mathrm{c}_{13} = 2.17 \times 10^{10} \mathrm{N.m}^{-2},$$
$$\mathrm{c}_{33} = 6.17 \times 10^{10} \mathrm{N.m}^{-2}, \mathrm{c}_{44} = 3.278 \times 10^{10} \mathrm{N.m}^{-2}, T_0 = .298 \times 10^3 K,$$
$$a_1 = 2.68 \times 10^6 Nm^{-2} K^{-1}, a_3 = 2.68 \times 10^6 Nm^{-2} K^{-1}, K_3 = 1.7 \times 10^2 Wm^{-1} K^{-1},$$
$$K_3 = 1.7 \times 10^2 Wm^{-1} K^{-1}, \alpha_{1c} = 2.1 \times 10^{-4} m^3.Kg^{-1}, \alpha_{3c} = 2.5 \times 10^{-4} m^3.Kg^{-1},$$
$$a = 2.4 \times 10^4 m^2 s^{-2} K^{-1}, b = 13 \times 10^5 Kgm^5 s^{-2}, \alpha_1^* = .95 \times 10^{-8} m^{-3}.s.Kg,$$
$$\alpha_3^* = .90 \times 10^{-8} m^{-3}.s.Kg.$$

Figures 2 to 5 depict the variations of radial displacement u_r, axial displacement u_z, temperature change T and

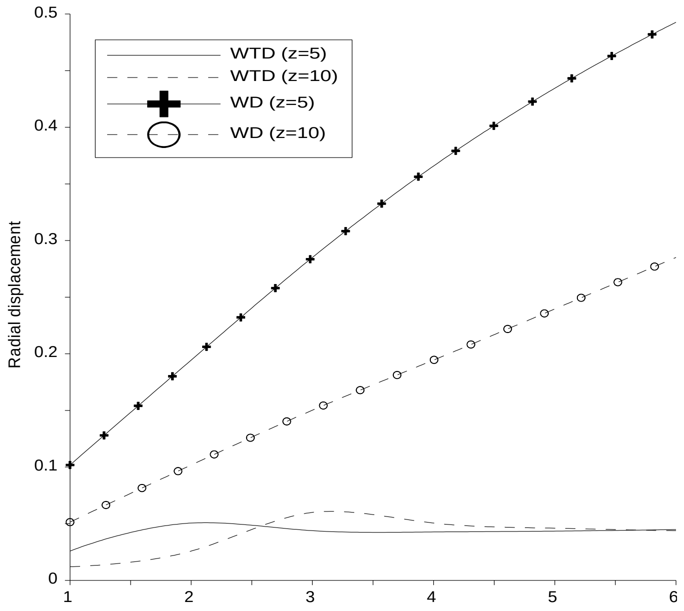

Figure 2. Variation of radial displacement u_r w.r.t. r.

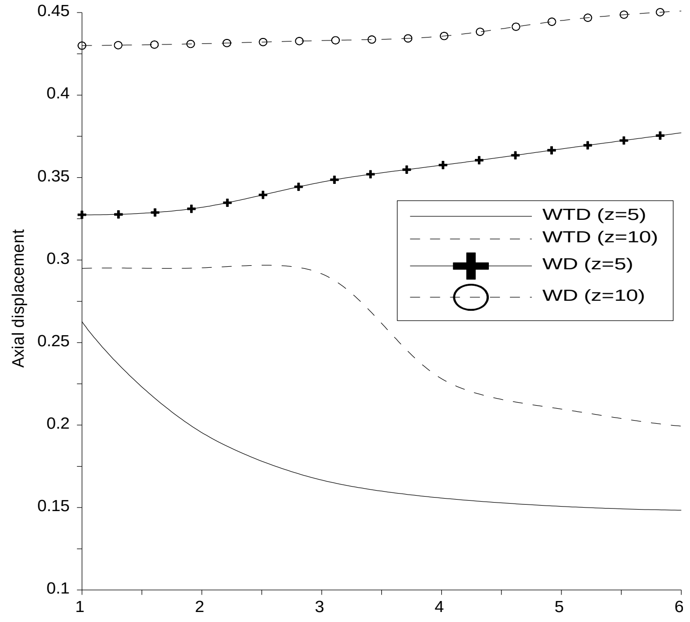

Figure 3. Variation of axial displacement u_θ w.r.t. r.

mass concentration C w.r.t. r for thermoelastic diffusion material. The solid and dotted line respectively, corresponds to thermoelastic theory (WTD$^{z=5}$), (WTD$^{z=10}$) and centre symbols on these lines, respectively corresponds to thermoelastic theory with mass diffusion (WD$^{z=5}$) ,(WD $^{z=10}$).

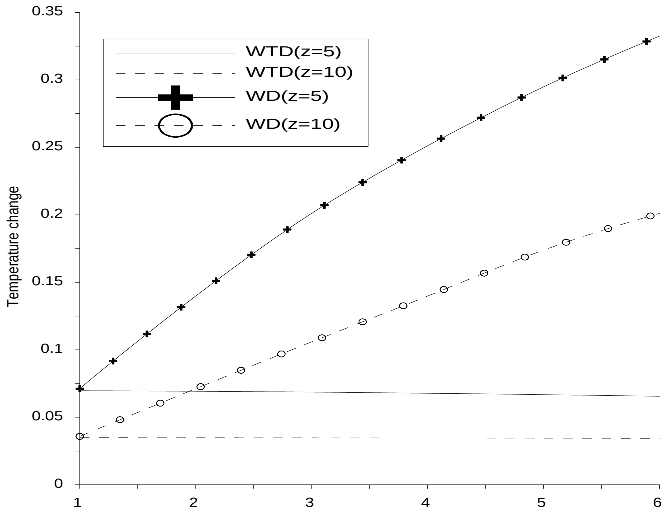

Figure 4. Variation of temperature distribution T w.r.t. r.

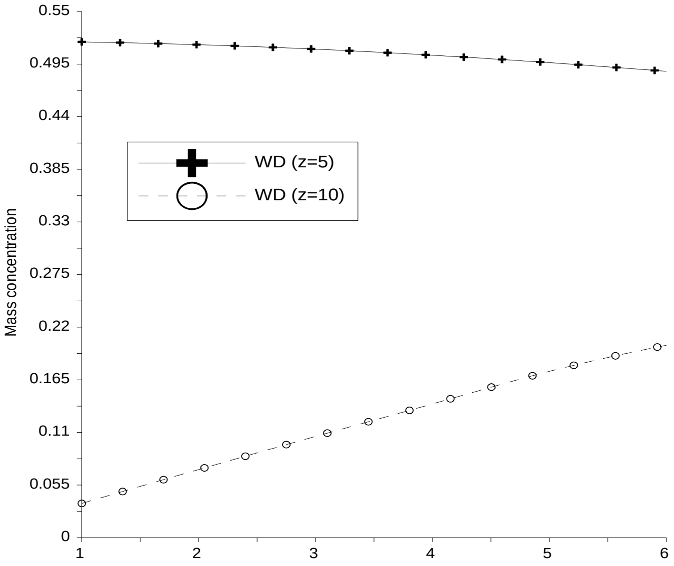

Figure 5. Variation of mass concentration distribution C w.r.t. r.

Figure 2 shows that the values of u_r in case of WTD slightly decrease for smaller values of r and for higher values of r, the values of u_r become dispersionless, although for the case of WD, the values of u_r increase for all values of r. It is noticed that the values of u_r in case of WD remain more in comparison with WTD. Figure 3 depicts that the values of u_z in case of WTD decrease for all values of r, whereas for the case of WD, the values of u_z slightly increase for smaller values of r and finally becomes constant.

It is evident that the values of u_z in case of WD remain more in comparison with WTD. Figure 4 shows that the values of T in case of WTD slightly decreases for all values of $r,$ although for the case of WD, the values of T increase for all values of $r.$ It is noticed that the values of T in case of WD remain more in comparison with WTD. Figure 5 depicts that the values of C in case of $z = 5$ slightly decrease for all values of $r,$ whereas for the case of $z = 10$ the values of C increases for all values of r. It is evident that the values of T in case of $z = 5$ remain more in comparison with $z = 10$.

Conclusion

The Green's functions for three-dimensional problem in transversely isotropic thermoelastic diffusion medium have been derived for static case. After applying the dimensionless quantities and using the operator theory, we have obtained the general expression for components of displacement, temperature distribution, mass concentration and stress components in Cartesian as well as in cylindrical coordinates. Based on the obtained general solution, the three- dimensional Green's function for a study point heat source on the apex of a transversely isotropic thermoelastic cone in case of steady state problem are derived by four newly introduced harmonic functions. All components of thermoelastic field are expressed in terms of elementary functions and are convenient to use.

From the present investigation, a special case of interest is deduced to depict the effect of diffusion. From numerical results, we conclude that the values of horizontal displacement $u_r,$ axial displacement u_z and temperature change T remain more in case of themoelastic diffusion (WD) in comparison to themoelastic medium (WTD).

Conflict of Interest

The authors have not declared any conflict of interest.

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Appendix A

$$\bar{a} = \delta_1(\gamma_1 q_4^* - \delta_5 q_8^*),\ \bar{b} = (\delta_4^2 - \delta_1^2)q_8^* + \delta_5(q_2^* - q_8^*) + q_4^*(\gamma_1 - \delta_4) + \delta_1(\gamma_1 q_3^* - \delta_5 q_7^*) - \delta_4 q_2^* \gamma_1,$$
$$c = (\delta_4^2 - \delta_1^2)q_7^* + q_3^*(\gamma_1 - \delta_4) - \varepsilon_1 q_2^* \delta_4 + \delta_5(q_1^* - q_7^*) + \delta_1(q_2^* - q_8^*),\ d = \delta_1(q_1^* - q_7^*),$$
$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Appendix B

$$\bar{a}_1 = (q_5^* - q_7^*)\delta_1,\ \bar{b}_1 = \delta_1(q_8^* - q_6^*) + \delta_5(q_5^* - q_7^*) + \varepsilon_1(\delta_4 q_7^* - q_3^*) - \gamma_1 \delta_4 q_5^*$$
$$\bar{c}_1 = (\gamma_1 q_6^* + \varepsilon_1 q_8^*)\delta_4 + (q_8^* - q_6^*)\delta_5 - q_4^* \varepsilon_1$$
$$\bar{a}_2 = (q_1^* + q_5^*)\gamma_1 + \varepsilon_1(q_1^* - q_7^*) + \delta_4(q_7^* - q_5^*),\ \bar{b}_2 = \delta_1(\gamma_1 q_5^* - \varepsilon_1 q_7^*) +$$
$$\varepsilon_1(q_2^* + q_8^*) - \gamma_1(q_2^* + q_6^*) + \delta_4(q_6^* - q_8^*), \bar{c}_2 = \delta_1(\varepsilon_1 q_8^* - \gamma_1 q_6^*),$$
$$\bar{a}_3 = (q_1^* - q_5^*)\delta_1,\ \bar{b}_3 = (\delta_4^2 - \delta_1^2)q_5^* + \delta_5(q_1^* - q_5^*) + \delta_1(q_2^* + q_6^*) - \delta_4 \varepsilon_1 q_1^*$$
$$\bar{c}_3 = (\delta_1^2 - \delta_4^2)q_6^* + \delta_5(q_2^* + q_6^*) - \delta_4(\varepsilon_1 q_2^* + 1) - \delta_1 \delta_5 q_5^*,\ \bar{d}_4 = \delta_1 \delta_5 q_6^*$$

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Multivalent harmonic uniformly convex functions

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In this paper, several properties of the multivalent harmonic uniformly convex classes $K_H(m,\alpha)$ and $\overline{K_H}(m,\alpha)$. were investigated. Coefficient bounds, distortion theorem, extreme points, convolution condition, convex combinations and integral operator for these classes were obtained.

Key words: Harmonic, multivalent functions, convex, convolution.

INTRODUCTION

A continuous complex valued function $f=u+iv$ which is defined in a simply connected complex domain *D* is said to be harmonic in D if both u and v are real harmonic in *D*. In any simply connected domain we can write:

$$f(z)=h(z)+\overline{g(z)}, \tag{1}$$

where *h* and *g* are analytic in *D*. We call *h* the analytic part and *g* the co-analytic part of *f*. A necessary and sufficient condition for *f* to be locally univalent and sense-preserving in D is that $|h'(z)|>|g'(z)|$ in *D* (Clunie and Sheil-Small, 1984).

Denote by S_H, the class of functions *f* of the form (2) that are harmonic univalent and sense preserving in the unit disc $U=\{z:|z|<1\}$ for which $f(0)=f_z(0)-1=0.$ For $f=h+\overline{g}\in S_H$, we may express:

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k+\sum_{k=1}^{\infty}\overline{b_k z^k},\ |b_1|<1, \tag{2}$$

where the analytic functions *h* and *g* are of the form:

$$h(z)=z+\sum_{k=2}^{\infty}a_k z^k,\quad g(z)=\sum_{k=2}^{\infty}b_k z^k,\ |b_1|<1. \tag{3}$$

Clunie and Sheil-Small (1984) investigated the class S_H as well as its geometric subclasses and some coefficient bounds for functions in S_H were obtained. Since then, various subclasses of S_H were investigated by several authors (Al-Shaqsi and Darus, 2008; Chandrashekar et al., 2009; Jahangiri, 1999; Murugusundaramoorthy, 2003; Murugusundaramoorthy et al., 2009; Rosy et al., 2001).

Recently, Kanas and Wisniowska (1999), Kanas and Srivastava (2000) studied the class of k -uniformly convex analytic functions. For $m\geq 1$ and $0\leq\alpha<1,$ we let

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$H(m)$ denote the class of multivalent harmonic functions $f(z)=h(z)+\overline{g(z)},$ where

$$h(z)=z^m+\sum_{k=m+1}^{\infty}a_k z^k,\ g(z)=\sum_{k=m}^{\infty}b_k z^k,\ |b_m|<1. \tag{4}$$

We consider the class $K_H(m,\alpha)$ of functions of the form (1) where *h* and *g* are given by Equation (4) satisfying the inequality

$$\operatorname{Re}\left(\frac{z^2h''(z)+zh'(z)+\overline{z^2g''(z)}+\overline{zg'(z)}}{zh'(z)-\overline{zg'(z)}}\right) \geq\left|\frac{z^2h''(z)+zh'(z)+\overline{z^2g''(z)}+\overline{zg'(z)}}{zh'(z)-\overline{zg'(z)}}-m\right|+m\alpha, \tag{5}$$

where $m\geq 1$ and $0\leq\alpha<1.$

Using the fact that $\operatorname{Re}(w)>|w-m|+m\alpha\Leftrightarrow\operatorname{Re}[(1+e^{i\varphi})w-me^{i\varphi}]\geq m\alpha,$ it follows from the condition (5) that *f* is in the class $K_H(m,\alpha)$ if and only if

$$\operatorname{Re}\left\{m+(1+e^{i\varphi})\left(\frac{z^2h''(z)+zh'(z)+\overline{z^2g''(z)}+\overline{zg'(z)}}{zh'(z)-\overline{zg'(z)}}-m\right)\right\}\geq m\alpha, \tag{6}$$

where $m\geq 1$ and $0\leq\alpha<1.$

We note that:

Putting $m=1,$ $K_H(1,\alpha)=HCV(1,\alpha)$ (Kim et al., 2002).

Further, let $\overline{K}_H(m,\alpha)$ be the subclass of $K_H(m,\alpha)$ consisting of functions of the form:

$$f(z)=z^m-\sum_{k=m+1}^{\infty}|a_k|z^k-\sum_{k=m}^{\infty}|b_k|\overline{z}^k\quad(|b_m|<1)\ . \tag{7}$$

Recent interest in the study of multivalent harmonic functions in the plan prompted the publication of several articles, such as Ahuja and Jahangiri (2001, 2002, 2003), Bshouty et al. (1999), Guney and Ahuja (2006).

In this paper, the coefficient bounds for the classes $K_H(m,\alpha)$ and $\overline{K}_H(m,\alpha)$. as well as distortion theorem, extreme points, convolution, convex combinations and integral operator for functions in the class $\overline{K}_H(m,\alpha)$ were obtained.

COEFFICIENTS BOUNDS AND DISTORTION THEOREM

Unless otherwise mentioned, it was assumed in the course of this study that $0\leq\alpha<1,\ m\geq 1$ and $z\in U$. We began with a sufficient condition for functions in the classes $K_H(m,\alpha)$ and $\overline{K}_H(m,\alpha)$ and obtained distortion theorem for functions in the class $\overline{K}_H(m,\alpha)$.

Theorem 1

Let $f=h+\overline{g},$ where *h* and *g* are given by Equation (4), and satisfy the condition

$$\sum_{k=m+1}^{\infty}\frac{k[2k-m(1+\alpha)]}{m(1-\alpha)+1-|m(1-\alpha)-1|}|a_k| +\sum_{k=m}^{\infty}\frac{k[2k+m(1+\alpha)]}{m(1-\alpha)+1-|m(1-\alpha)-1|}|b_k|\leq\frac{1}{2}\ . \tag{8}$$

Then $f(z)\in K_H(m,\alpha).$

Proof

Assume that Equation (8) holds. It suffices to prove that

$$\operatorname{Re}\left\{m+(1+e^{i\varphi})\left(\frac{z^2h''(z)+zh'(z)+\overline{z^2g''(z)}+\overline{zg'(z)}}{zh'(z)-\overline{zg'(z)}}-m\right)-m\alpha\right\} =\operatorname{Re}\frac{A(z)}{B(z)}\geq 0. \tag{9}$$

Using the fact that $\operatorname{Re}\{w\}\geq 0$ if and only if $|1+w|\geq|1-w|$, it suffices to show that

$$|A(z)+B(z)|-|A(z)-B(z)|\geq 0, \tag{10}$$

where

$$A(z)=[(1-m)\left(1+e^{i\varphi}\right)+m(1-\alpha)]zh'(z)+(1+e^{i\varphi})z^2h''(z)$$

$$+[(1+m)\left(1+e^{i\varphi}\right)-m(1-\alpha)]\overline{zg'(z)}+(1+e^{i\varphi})\overline{z^2g''(z)}$$

and

$$B(z)=zh'(z)-\overline{zg'(z)}.$$

Substituting for *A(z)* and *B(z)* in the left side of Equation (10) we obtain:

$$|A(z)+B(z)|-|A(z)-B(z)|$$

$$=\left|m[1+m(1-\alpha)]z^m+\sum_{k=m+1}^{\infty}k[k-m\alpha+(k-m)e^{i\varphi}+1]a_k z^k\right.$$

$$+\sum_{k=m}^{\infty} k[k+m\alpha+(k+m)e^{i\varphi}-1]\overline{b_k z^k}\ \Big|$$

$$-\Big|m[-1+m(1-\alpha)]z^m+\sum_{k=m+1}^{\infty} k[k-m\alpha+(k-m)e^{i\varphi}-1]a_k z^k$$

$$+\sum_{k=m}^{\infty} k[k+m\alpha+(k+m)e^{i\varphi}+1]\overline{b_k z^k}\ \Big|$$

$$\geq m[m(1-\alpha)+1)]|z|^m-\sum_{k=m+1}^{\infty} k[2k-m(1+\alpha)+1]|a_k||z|^k$$

$$-\sum_{k=m+1}^{\infty} k[2k+m(1+\alpha)-1]|b_k||z|^k$$

$$-|[m(m(1-\alpha)-1)]||z|^m-\sum_{k=m+1}^{\infty} k[2k-m(1+\alpha)-1]|a_k||z|^k$$

$$-\sum_{k=m}^{\infty} k[2k+m(1+\alpha)+1]|b_k||z|^k$$

$$\geq m[(m(1-\alpha)+1)-|(m(1-\alpha)-1)|]|z|^m\cdot$$

$$\cdot\left\{1-\sum_{k=m+1}^{\infty}\frac{2k[2k-m(1+\alpha)]}{m[(m(1-\alpha)+1)-|(m(1-\alpha)-1)|]}|a_k||z|^{k-m}\right.$$

$$\left.-\sum_{k=m}^{\infty}\frac{2k[2k+m(1+\alpha)]}{m[(m(1-\alpha)+1)-|(m(1-\alpha)-1)|]}|a_k||z|^{k-m}\right\}$$

$$\geq m[m(1-\alpha)+1)-|(m(1-\alpha)-1)|]\cdot$$

$$\cdot\left\{1-\sum_{k=m+1}^{\infty}\frac{2k[2k-m(1+\alpha)]}{m[(m(1-\alpha)+1)-|(m(1-\alpha)-1)|]}|a_k|\right.$$

$$\left.-\sum_{k=m}^{\infty}\frac{2k[2k+m(1+\alpha)]}{m[(m(1-\alpha)+1)-|(m(1-\alpha)-1)|]}|b_k|\right\} \geq 0,$$

using Equation (8).

The functions

$$f(z)=z^m+\sum_{k=m+1}^{\infty}\frac{m[(m(1-\alpha)+1)-|(m(1-\alpha)-1)|]}{k[2k-m(1+\alpha)]}x_k z^k$$

$$+\sum_{k=m}^{\infty}\frac{m[(m(1-\alpha)+1)-|(m(1-\alpha)-1)|]}{k[2k+m(1+\alpha)]}\overline{y_k z^k}, \tag{11}$$

where $\sum_{k=m+1}^{\infty}|x_k|+\sum_{k=m}^{\infty}|y_k|=1,$ shows that the coefficient bound given by Equation (8) is sharp. This completes the proof of Theorem 1.

Corollary 1

Let $f=h+\overline{g},$ where h and g are given by Equation (4). Also, let $1\leq m\leq 1/(1-\alpha)$ and if the condition:

$$\sum_{k=m+1}^{\infty}\frac{k[2k-m(1+\alpha)]}{m^2(1-\alpha)}|a_k|+\sum_{k=m}^{\infty}\frac{k[2k+m(1+\alpha)]}{m^2(1-\alpha)}|b_k|\leq 1, \tag{12}$$

holds, then $f(z)\in K_H(m,\alpha).$

Corollary 2

Let $f=h+\overline{g},$ where h and g are given by Equation (4). Also, let $m\geq 1/(1-\alpha)$ and if the condition

$$\sum_{k=m+1}^{\infty}k[2k-m(1+\alpha)]|a_k|+\sum_{k=m}^{\infty}k[2k+m(1+\alpha)]|b_k|\leq 1, \tag{13}$$

holds, then $f(z)\in K_H(m,\alpha).$

In the following theorem, it is shown that the condition (12) is also necessary for function $f=h+g$, where f is of the form (7).

Theorem 2

Let $f=h+\overline{g},$ be given by the form (7). Then $f(z)\in\overline{K}_H(m,\alpha),$ if and only if the coefficient bound (12) holds.

Proof

Since $\overline{K}_H(m,\alpha)\subseteq K_H(m,\alpha),$ we only need to prove this part of the theorem. To this end, for functions $\overline{K}_H(m,\alpha),$ it was noticed that the necessary and sufficient condition to be in the class $\overline{K}_H(m,\alpha)$ is that:

$$\operatorname{Re}\left\{m+(1+e^{i\varphi})\left(\frac{z^2h''(z)+zh'(z)+\overline{z^2g''(z)}+\overline{zg'(z)}}{zh'(z)-\overline{zg'(z)}}-m\right)-m\alpha\right\}\geq 0. \tag{14}$$

This is equivalent to

$$\operatorname{Re}\left\{\frac{m^2(1-\alpha)z^m-\sum_{k=m+1}^{\infty}k[2k-m(1+\alpha)]|a_k|z^k-\sum_{k=m}^{\infty}k[2k+m(1+\alpha)]|b_k|\overline{z}^k}{z^m-\sum_{k=m+1}^{\infty}k|a_k|z^k+\sum_{k=m}^{\infty}k|b_k|\overline{z}^k}\right\}\geq 0.$$

This condition must hold for all values of $z\in U$ and for real α so that on taking $z=r<1$, the above inequality reduces to:

$$\frac{1-\sum_{k=m+1}^{\infty}\frac{k[2k-m(1+\alpha)]}{m^2(1-\alpha)}|a_k|r^{k-m}-\sum_{k=m}^{\infty}\frac{k[2k+m(1+\alpha)]}{m^2(1-\alpha)}|b_k|r^{k-m}}{1-\sum_{k=m+1}^{\infty}k|a_k|r^{k-m}+\sum_{k=m}^{\infty}k|b_k|r^{k-m}}\geq 0. \tag{15}$$

Letting $r\to 1^-$ through real values, we obtain the condition (12). This completes the proof of Theorem 2.

Theorem 3

Let the function $f(z)$ given by Equation (7) be in the class $\overline{K}_H(m,\alpha),$ then for $z = r < 1$

$$|f(z)| \le \begin{cases} (1+|b_m|)r^m + \frac{1}{m+1}\left(\frac{m^2(1-\alpha)}{2+m(1-\alpha)} - \frac{m^2(3+\alpha)}{2+m(1-\alpha)}|b_m|\right)r^{m+1} & , m(1-\alpha)\le 1 \\ (1+|b_m|)r^m + \frac{1}{m+1}\left(\frac{1}{2+m(1-\alpha)} - \frac{m^2(3+\alpha)}{2+m(1-\alpha)}|b_m|\right)r^{m+1} & , m(1-\alpha)\ge 1, \end{cases} \tag{16}$$

and

$$|f(z)| \ge \begin{cases} (1-|b_m|)r^m - \frac{1}{m+1}\left(\frac{m^2(1-\alpha)}{2+m(1-\alpha)} - \frac{m^2(3+\alpha)}{2+m(1-\alpha)}|b_m|\right)r^{m+1} & , m(1-\alpha)\le 1 \\ (1-|b_m|)r^m - \frac{1}{m+1}\left(\frac{m^2(1-\alpha)}{2+m(1-\alpha)} - \frac{m^2(3+\alpha)}{2+m(1-\alpha)}|b_m|\right)r^{m+1} & , m(1-\alpha)\ge 1, \end{cases} \tag{17}$$

where $|b_m| \le \dfrac{(1-\alpha)}{(3+\alpha)}.$

Proof

If $m(1-\alpha)\le 1,$ we have,

$$|f(z)| \le (1+|b_m|)r^m + \sum_{k=m+1}^{\infty}(|a_k|+|b_k|)r^k$$

$$\le (1+|b_m|)r^m + \sum_{k=m+1}^{\infty}(|a_k|+|b_k|)r^{m+1}$$

$$\le (1+|b_m|)r^m + \frac{m^2(1-\alpha)}{(m+1)[2+m(1-\alpha)]}\sum_{k=m+1}^{\infty}(m+1)\left(\frac{2+m(1-\alpha)}{m^2(1-\alpha)}\right)(|a_k|+|b_k|)r^{m+1}$$

$$\le (1+|b_m|)r^m + \frac{m^2(1-\alpha)}{(m+1)[2+m(1-\alpha)]}\left[\sum_{k=m+1}^{\infty}\left(\frac{k[2k-m(1+\alpha)]}{m^2(1-\alpha)}\right)|a_k| + \left(\frac{k[2k+m(1+\alpha)]}{m^2(1-\alpha)}\right)|b_k|\right]r^{m+1}$$

$$\le (1+|b_m|)r^m + \frac{m^2(1-\alpha)}{(m+1)[2+m(1-\alpha)]}\left[1-\frac{(3+\alpha)}{(1-\alpha)}\right]r^{m+1}$$

$$= (1+|b_m|)r^m + \frac{1}{m+1}\left(\frac{m^2(1-\alpha)}{2+m(1-\alpha)} - \frac{m^2(3+\alpha)}{2+m(1-\alpha)}|b_m|\right)r^{m+1},$$

which proves the assertion (16) of Theorem 3. The proof of the assertion (17) is similar, thus, it was omitted.

Remark 1

Putting $m=1$ in Theorem 3, we improve the result obtained by Kim et al. (2002) by adding the condition $|b_1| \le \frac{(1-\alpha)}{(3+\alpha)}.$

The following covering result follows the left hand inequality Theorem 3.

Corollary 3

Let the function $f(z)$ given by (7) be in the class $\overline{K}_H(m,\alpha),$ then for $|z| = r < 1,$ we have:

$$\left\{ w : |w| < \begin{bmatrix} \frac{2+m(3-\alpha)}{(m+1)(2+m(1-\alpha)} - \frac{2+m(3-2m-\alpha(2m+1)}{(m+1)(2+m(1-\alpha)}|b_m| & , m(1-\alpha)\le 1 \\ \frac{1+m[3+m(1-\alpha)-\alpha]}{(m+1)(2+m(1-\alpha)} - \frac{2+m(3-2m-\alpha(2m+1)}{(m+1)(2+m(1-\alpha)}|b_m| & , m(1-\alpha)\ge 1 \end{bmatrix} \right\} \subset f(U),$$

where $|b_m| < \dfrac{2+m(3-\alpha)}{2+m(3-2m-\alpha(2m+1)},$ or $|b_m| < \dfrac{1+m[3+m(1-\alpha)-\alpha]}{2+m(3-2m-\alpha(2m+1)}.$

EXTREME POINTS

Here, the extreme points of the closed convex hull of the class $\overline{K}_H(m,\alpha)$ denoted by $clco\,\overline{K}_H(m,\alpha).$ was determined

Theorem 4

Let *f(z)* be given by (7), then $f(z)\in clco\,\overline{K}_H(m,\alpha)$ if and only if

$$f(z) = \sum_{k=m}^{\infty}\left[x_k h_k(z) + y_k g_k(z)\right], \tag{18}$$

where

$$h_m(z) = z^m,$$

$$h_k(z) = \begin{cases} z^m - \frac{m^2(1-\alpha)}{k[2k-m(1+\alpha)]}z^k \ (k\ge m+1) & , m(1-\alpha)\le 1 \\ z^m - \frac{1}{k[2k-m(1+\alpha)]}z^k \ (k\ge m+1) & , m(1-\alpha)\ge 1, \end{cases}$$

And

$$g_k(z) = \begin{cases} z^m - \frac{m^2(1-\alpha)}{k[2k+m(1+\alpha)]}\overline{z}^k \ (k\ge m) & , m(1-\alpha)\le 1 \\ z^m - \frac{1}{k[2k+m(1+\alpha)]}\overline{z}^k \ (k\ge m) & , m(1-\alpha)\ge 1, \end{cases}$$

Where $\sum_{k=m}^{\infty}(x_k+y_k)=1,\ x_k\ge 0$ and $y_k\ge 0.$

In particular, the extreme points of the class $\overline{K}_H(m,\alpha)$ are $\{h_k\}(k\ge m+1)$ and $\{g_k\}(k\ge m),$ respectively.

Proof

For a function *f(z)* of the form (18), we have:

$$f(z)=\sum_{k=m}^{\infty}\left[x_k h_k(z)+y_k g_k(z)\right]$$

$$=\sum_{k=m}^{\infty} x_k\left(z^m-\frac{m^2(1-\alpha)}{k[2k-m(1+\alpha)]}z^k\right)+y_k\left(z^m-\frac{m^2(1-\alpha)}{k[2k+m(1+\alpha)]}\overline{z}^k\right)$$

$$=z^m-\sum_{k=m+1}^{\infty}\frac{m^2(1-\alpha)}{k[2k-m(1+\alpha)]}x_k z^k-\sum_{k=m}^{\infty}\frac{m^2(1-\alpha)}{k[2k+m(1+\alpha)]}y_k\overline{z}^k$$

but,

$$\sum_{k=m+1}^{\infty}\frac{k[2k-m(1+\alpha)]}{m^2(1-\alpha)}|a_k|+\sum_{k=m}^{\infty}\frac{k[2k+m(1+\alpha)]}{m^2(1-\alpha)}|b_k|$$

$$=\sum_{k=m+1}^{\infty}x_k+\sum_{k=m}^{\infty}y_k=1-x_k\le 1,$$

and $f(z)\in clco\,\overline{K}_H(m,\alpha).$

Conversely, assume that $f(z)\in clco\,\overline{K}_H(m,\alpha).$ Then

$$a_k=\frac{m^2(1-\alpha)}{k[2k-m(1+\alpha)]},$$

and

$$b_k=\frac{m^2(1-\alpha)}{k[2k+m(1+\alpha)]}$$

set

$$x_k=\frac{k[2k-m(1+\alpha)]}{m^2(1-\alpha)}|a_k|,$$

and

$$y_k=\frac{k[2k+m(1+\alpha)]}{m^2(1-\alpha)}|b_k|$$

Then by using Equation (12), we have $0\le x_k\le 1(k=m+1,m+2,...)$ and $0\le y_k\le 1(k=m,m+1,...).$

$x_m=1-\sum_{k=m+1}^{\infty}x_k-\sum_{k=m}^{\infty}y_k.$ is defined and the equation: $f(z)=\sum_{k=m}^{\infty}(x_k h_k+y_k g_k).$ is obtained. This completes the proof of Theorem 4.

CONVOLUTION AND CONVEX COMBINATION

In this section, the convolution properties and convex combination were determined.

Let the functions $f_j(z)$ be defined by:

$$f_j(z)=z^m-\sum_{k=m+1}^{\infty}\left|a_{k,j}\right|z^k-\sum_{k=m}^{\infty}\left|b_{k,j}\right|\overline{z}^k\ (j=1,2),\tag{19}$$

be in the class $\overline{K}_H(m,\alpha)$, we denote by $(f_1*f_2)(z)$ the convolution or (Hadamard Product) of the function $f_1(z)$ and $f_2(z)$, that is,

$$(f_1*f_2)(z)=z^m-\sum_{k=m+1}^{\infty}\left|a_{k,1}\right|\left|a_{k,2}\right|z^k-\sum_{k=m}^{\infty}\left|b_{k,1}\right|\left|b_{k,2}\right|\overline{z}^k.\tag{20}$$

while the integral convolution is defined by

$$(f_1\Diamond f_2)(z)=z^m-\sum_{k=m+1}^{\infty}\frac{m\left|a_{k,1}\right|\left|a_{k,2}\right|}{k}z^k-\sum_{k=m}^{\infty}\frac{m\left|b_{k,1}\right|\left|b_{k,2}\right|}{k}\overline{z}^k.\tag{21}$$

We first show that the class $\overline{K}_H(m,\alpha)$ is closed under convolution.

Theorem 5

For $0\le\delta\le\alpha<1$, let the functions $f_1(z)\in\overline{K}_H(m,\alpha)$ and $f_2(z)\in\overline{K}_H(m,\delta).$

Then

$$(f_1*f_2)(z)\in\overline{K}_H(m,\alpha)\subset\overline{K}_H(m,\delta),\tag{22}$$

$$(f_1\Diamond f_2)(z)\in\overline{K}_H(m,\alpha)\subset\overline{K}_H(m,\delta).\tag{23}$$

Proof

Let $f_j(z)(j=1,2)$ given by Equation (19), where $f_1(z)$ is in the class $\overline{K}_H(m,\alpha)$ and $f_2(z)$ be in the class $\overline{K}_H(m,\delta).$ It therefore shows that the coefficients of $(f_1*f_2)(z)$ satisfy the required condition given in Equation (12).

For $f_2(z)\in\overline{K}_H(m,\delta)$, we note that $\left|a_{k,2}\right|<1$ and $\left|b_{k,2}\right|<1$. Now for the convolution functions $(f_1*f_2)(z),$ we obtain

$$\sum_{k=m+1}^{\infty}\frac{k[2k-m(1+\delta)]}{m^2(1-\delta)}\left|a_{k,1}\right|\left|a_{k,2}\right|+\sum_{k=m}^{\infty}\frac{k[2k+m(1+\delta)]}{m^2(1-\delta)}\left|b_{k,1}\right|\left|b_{k,2}\right|$$

$$\le\sum_{k=m+1}^{\infty}\frac{k[2k-m(1+\delta)]}{m^2(1-\delta)}\left|a_{k,1}\right|+\sum_{k=m}^{\infty}\frac{k[2k+m(1+\delta)]}{m^2(1-\delta)}\left|b_{k,1}\right|$$

$$\le\sum_{k=m+1}^{\infty}\frac{k[2k-m(1+\alpha)]}{m^2(1-\alpha)}\left|a_{k,1}\right|+\sum_{k=m}^{\infty}\frac{k[2k+m(1+\alpha)]}{m^2(1-\alpha)}\left|b_{k,1}\right|\le 1,$$

since $0 \le \delta \le \alpha < 1$ and $f_1(z) \in \overline{K}_H(m,\alpha).$

Thus $(f_1 * f_2)(z) \in \overline{K}_H(m,\alpha) \subset \overline{K}_H(m,\delta).$ The proof of the assertion (23) is similar, thus, it was omitted. This completes the proof of Theorem 5.

Next we show that $\overline{K}_H(m,\alpha)$ is closed under convex combinations of its members.

Theorem 6

The class $\overline{K}_H(m,\alpha)$ is closed under convex combination.

Proof

For $i = 1, 2, \ldots$, let $f_i(z) \in \overline{K}_H(m,\alpha)$, where

$$f_i(z) = z^m - \sum_{k=m+1}^{\infty} \left|a_{k,i}\right| z^k - \sum_{k=m}^{\infty} \left|b_{k,i}\right| \overline{z}^k \quad (z \in U; i = 1, 2, \ldots), \tag{24}$$

then from (12), for $\sum_{i=1}^{\infty} t_i = 1,\ 0 \le t_i < 1$, the convex combination of $f_i(z)$ may be written as:

$$\sum_{i=1}^{\infty} t_i f_i(z) = z^m - \sum_{k=m+1}^{\infty} \left(\sum_{i=1}^{\infty} t_i \left|a_{k,i}\right| \right) z^k - \sum_{k=m}^{\infty} \left(\sum_{i=1}^{\infty} t_i \left|b_{k,i}\right| \right) \overline{z}^k. \tag{25}$$

Then by using Equation (12), we have

$$\begin{aligned}
&\sum_{k=m+1}^{\infty} \frac{k[2k - m(1+\alpha)]}{m^2(1-\alpha)} \left(\sum_{i=1}^{\infty} t_i \left|a_{k,i}\right| \right) + \sum_{k=m}^{\infty} \frac{k[2k - m(1+\alpha)]}{m^2(1-\alpha)} \left(\sum_{i=1}^{\infty} t_i \left|b_{k,i}\right| \right) \\
&= \sum_{i=1}^{\infty} t_i \left[\sum_{k=m+1}^{\infty} \frac{k[2k - m(1+\alpha)]}{m^2(1-\alpha)} \left|a_{k,i}\right| + \sum_{k=m}^{\infty} \frac{k[2k - m(1+\alpha)]}{m^2(1-\alpha)} \left|b_{k,i}\right| \right] \\
&\le \sum_{i=1}^{\infty} t_i \le 1.
\end{aligned}$$

This completes the proof of Theorem 6.

Integral operator

Here, a closure property of the class $\overline{K}_H(m,\alpha)$ was examined under the generalized Bernardi-Libera-Livingston integral operator (Saitoh et al., 1992), $L_{c,m}(f(z))$ which is defined by:

$$L_{c,m}(f(z)) = \left(\frac{c+m}{z^c} \right) \int_0^z t^{c-1} f(t)\,dt,\ c > -m. \tag{26}$$

Theorem 7

Let $\overline{K}_H(m,\alpha).$

Then

$$L_{c,m}(f(z)) \in \overline{K}_H(m,\alpha).$$

Proof

From Equation (26), it follows that

$$\begin{aligned}
L_{c,m}(f(z)) &= \left(\frac{c+m}{z^c} \right) \int_0^z t^{c-1} \left[h(t) + \overline{g(t)} \right] dt \\
&= \left(\frac{c+m}{z^c} \right) \left[\int_0^z t^{c-1} \left(t^m - \sum_{k=m+1}^{\infty} a_k t^k \right) dt - \int_0^z \overline{(t^{c-1} \sum_{k=m}^{\infty} b_k t^k)}\, dt \right] \\
&= z^m - \sum_{k=m+1}^{\infty} A_k z^k - \sum_{k=m}^{\infty} B_k z^k,
\end{aligned}$$

Where

$$A_k = \left(\frac{c+m}{c+k} \right) a_k,\ B_k = \left(\frac{c+m}{c+k} \right) b_k.$$

Therefore,

$$\begin{aligned}
&\sum_{k=m+1}^{\infty} \frac{k[2k - m(1+\alpha)]}{m^2(1-\alpha)} \left(\frac{c+m}{c+k} \right) |a_k| + \sum_{k=m}^{\infty} \frac{k[2k + m(1+\alpha)]}{m^2(1-\alpha)} \left(\frac{c+m}{c+k} \right) |b_k| \\
&\le \sum_{k=m+1}^{\infty} \frac{k[2k - m(1+\alpha)]}{m^2(1-\alpha)} |a_k| + \sum_{k=m}^{\infty} \frac{k[2k + m(1+\alpha)]}{m^2(1-\alpha)} |b_k| \le 1.
\end{aligned}$$

Since $f(z) \in \overline{K}_H(m,\alpha),$ by using Corollary 1, then $L_{c,m}(f(z)) \in \overline{K}_H(m,\alpha).$

This completes the proof of Theorem 7.

Remark 2

Putting $m = 1$ in the above results, the corresponding results by Kim et al. (2002), with $k = 1$ is obtained.

Conflict of Interest

The authors have not declared any conflict of interest.

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